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July 31, 2023

<u>***Via Email:***</u>
ATTN: MO MADANI
Florida Building Commission
Product Oversight Committee
2601 Blair Stone Road
Tallahassee, Florida 32399
Mo.madani@myfloridalicense.com

> **Re: RESPONSE TO YIPING WANG COMPLAINT RE: FL #15709.11**

Dear Product Oversight Committee,

Please be advised this firm represents, WinDoor Incorporated ("WinDoor"). Accordingly, please direct any and all correspondence regarding this matter to the undersigned herein. Further, this correspondence is intended to serve as a formal written response to the Florida Department of Business & Professional Regulation Complaint submitted by Ms. Yiping Wang ("Ms. Wang"), dated May 18, 2023, and scheduled to be heard by the Product Oversight Committee (the "Committee") on August 8, 2023.

Not only has WinDoor satisfied all requirements of the Florida Building Code but it has also gone above and beyond to ensure its product is in compliance with the same; including but not limited to engaging Florida licensed Professional Engineers, corresponding with experts on ASTM E06.52 glass subcommittee—whom develop the E1300 standard, satisfying TAS 201, 202, and 203 testing, and conducting varying calculations, each with consistent results. Therefore, any such complaint to the contrary is unfounded, and not rooted in evidence.

First, Ms. Wang has brought a complaint on behalf of a competitor of WinDoor against WinDoor's product approval No. 15709.11 (the "Product") and brazenly asserts the Product's exterior glazing system is a safety hazard, despite the fact that the product passed all TAS Testing. She presents no evidence of a safety issue. Notably, Ms. Wang relies upon the standards set forth in ASTM E1300-16(6), which only covers single and double laminated glass. The Product, which she refers to, however, is triple laminated glass and therefore governed not by ASTM E1300-16(6), as utilized by Ms. Wang, but, rather, ASTM E1300-X9 ("X9"). X9 provides the requisite standards and proper calculations for laminated glass not otherwise covered by the non-factored load charts.

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WinDoor opted to utilize these standards in seeking approval for its triple laminated glass product and strictly followed the code in doing so.

Ms. Wang baldly and falsely asserts WinDoor must either (1) provide its product calculations, which demonstrate compliance or (2) revise the product approval to make them comply with the Florida Building Code. The Florida Building Code/Product Approval mechanisms provided by the Commission, however, do not require such a remedy.

The Florida Department of business and Professional Regulations sets forth the proper mechanisms by which the Florida Building Commission may regulate Product Approval. The same can be located in Chapter 61G20-3 of the Florida Building Commission. Specifically, Section 61G20-3.005, titled *Product Evaluation and Quality Assurance for State Approval*, supplies a product may be determined in compliance with the Code in the following manner:

> (d) An evaluation report from a Florida Registered Architect *or a licensed Florida Professional Engineer* developed and signed and sealed, based upon *standard tests* or standard comparative *or rational analysis, or a combination thereof* and indicates that the product was evaluated to be in compliance with the Code and that the product is, for the purpose intended, at least equivalent to that required by the Code.

FBC 61G20-3.005(1)(d) (emphasis added). Additionally, Section 61G20-3.006 provides a manufacturer may achieve product validation by an approved validation entity for state approval by means of technical validation as provided below:

> (2) Technical Validation.
> (a) Technical Validation of an Evaluation Report: Validation of compliance with the Code using an evaluation report from an approved evaluation entity that is not an independent third-party from the manufacturer shall be performed by an approved validation entity using the validation steps of the Administrative Validation and the following steps:
> 1. Determination that the evaluator has complied with acceptable standards of engineering principles.
> 2. Engineering verification that the evaluation complies with the Code.
> 3. Copy of the application complying with all aspects of Chapter 61G15-36, F.A.C., must be filed with the Commission.

The Product was evaluated by Professional Engineer, Steve Urich, and determined to be in compliance with all applicable standards of engineering principles and the Code.

Section 2404.4 of the 2020 Florida Building Code states, "the load resistance of glass under uniform load shall be determined in accordance with ASTM E1300." Moreover, it also states, "For designs outside the scope of this section, an analysis _or_ test data for the specific installation shall be prepared by a registered design professional." While WinDoor was only required to have provided either analysis _or_ test data, WinDoor, in fact, provided both analysis and test data in its application for the Product.

Even further, the Product was tested by a Miami-Dade certified laboratory to TAS 201, 202, and 203 standards, which was witnessed by a licensed Florida Professional Engineer. As a result, the Product passed all requirements of TAS 201, 202, and 203 standards, in accordance with Section 1609.1.2 and 1626 of the Florida Building Code, 7th Ed. (2020). Therefore, any and all complaints regarding the safety of the Product are unfounded and baseless.

While Ms. Wang provided calculations to the Committee in accordance with standards set forth in E1300-16(6), *the same are inapplicable to the Product in question*. The E1300-16(6) standard applies solely to single or double paned laminated glass. The Product is triple laminated glass. Thus, as provided in E1300-X9, wherein a product is not covered by the non-factored load charts, one may utilize Section E1300-X9 in performing engineering analysis of laminated glass. Accordingly, the manufacturer utilized Section E1300-X9 standards in developing its triple laminated glass product—as is specifically provided for within the Code.

Moreover, the manufacturer's licensed Florida Professional Engineer, consulted glazing experts on the ASTM E06.52 glass subcommittee, in the development of the Product. Notably, the ASTM E06.52 glass subcommittee, develops the E1300 standard. As a result of the manufacturer's inquiry regarding the use of the E1300-X9 standard for triple-laminated glass, one of the glazing experts provided the same was a proper method for "extending the E1300 method to 3 plies." He further, provided literature entitled, "Enhanced Effective Thickness of Multilayered Laminated Glass," which directly addressed three (3) plies. **Attached hereto as Exhibit A.** Subsequently the manufacturer's licensed Florida Professional Engineer performed additional calculations in accordance with the same; the results of which were consistent with those of the E1300-X9 calculations.

Finally, nowhere in Section 61G20-3, on Product Approval, or otherwise, provides an evaluation engineer is required to submit his/her calculations directly to the Florida Building Commission. Rather, the contrary is true. As provided in Section 61G20-3006(3), merely the validation entity is required to retain all copies of the requisite calculations for the time period as prescribed by statute or for the period the product remains approved, whichever is longer. Further, to achieve approval, a manufacturer needs to solely comply with the procedures outlined in Section 61G20-3.007; i.e. complete the requisite form (61G20-3.011(2)); validated as set forth in section 3.006, and submit the necessary fees.

Accordingly, it is plainly untrue that WinDoor needs to provide the Committee with a copy of its calculations. In fact, its calculations are a trade secret and their disclosure is specifically exempted from Florida public records laws. *See Managed Care of N. Am. Inc. v. Fla. Healthy Kids Corp.*, 268 So. 3d 856, 858 (Fla. 1st DCA 2019); *see also* Fla. Stat. § 815.045 ("*The Legislature finds that it is a public necessity that trade secret information as defined in s. 812.081 be expressly made confidential and exempt from the public records law because it is a felony to disclose such records*.") (emphasis added). The disclosure of trade secrets creates the potential for irreparable harm, and should be avoided. *See Superior Ins. Co. v. Cano*, 829 So. 2d 991 (Fla. 2d DCA 2002). Because WinDoor has a responsibility to protect its trade secrets and prevent their public disclosure, WinDoor is necessarily under no such obligation to disclose the same.

My client firmly denies any and all allegations contained in the Complaint and asks that the Committee dismisses same categorically. Should you have any questions regarding the foregoing or need any further documentation, please feel free to contact my office. Thank you in advance for your prompt attention to this matter.

Very truly yours,

Daniel J. DeLeo, Esq.

DJD/rkp
Enclosures

Enhanced Effective Thickness of Multi-Layered Laminated Glass

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Abstract

The stiffness and strength of laminated glass, a composite of glass layers bonded together by polymeric interlayers, depend upon shear coupling between the glass plies through the polymer. In the design practice, this effect is commonly considered by defining the effective thickness, i.e., the thickness of a monolith with equivalent bending properties. Traditional formulations have been proposed for a package of two layers of glass and one polymeric interlayer, but their extrapolation to a higher number of layers gives in general inaccurate results. Here, the recently-proposed *Enhanced Effective Thickness* method is extended to the case of laminated glass beams composed *i*) by three layers of glass of arbitrary thickness, or *ii*) by an arbitrary number of equally-thick glass layers. Comparison with numerical experiments confirms the accuracy of the proposed approach.

1

1   Introduction

Laminated glass is a composite widely used in civil engineering, as well as in automotive, aeronautics and shipbuilding. It is typically made of two glass plies bonded by a thermoplastic polymeric interlayer with a treatment in autoclave at high pressure and temperature. This process induces a strong chemical bond between materials, due to the union between hydroxyl groups along the polymer and silanol groups on the glass surface. Through lamination, safety in the post-glass-breakage phase is increased because fragments remain attached to the interlayer; risk of injuries is reduced and broken glass maintains a certain cohesion that prevents catastrophic collapse.

In the pre-glass-breakage phase, the polymeric interlayers are too soft to present flexural stiffness *per se*, but they can provide shear stresses that constrain the relative sliding of the glass plies ([1]). The precise calculation of this coupling is quite difficult and usually requires numerical analysis, complicated by the fact that response of the polymer is nonlinear, viscoelastic and temperature dependent. A common practice is to consider the polymer as linear elastic accounting for its viscoelasticity through an equivalent elastic modulus, assumed equal to the relaxed modulus under constant strain after a time comparable to the duration of the design action. Of course, the degree of coupling of the glass layers depends upon the shear stiffness of the interlayer ([2]). Thus, the flexural performance is somehow intermediate between the two borderline cases ([3], [4]) of *i) monolithic limit*, with perfect bonding between glass plies (shear-rigid interlayers) and *ii) layered limit*, with frictionless sliding glass plies. Since stress and strain are much lower in the monolithic than in the layered limit, to avoid redundant design a large number of studies, also in recent years,

have considered this subject ([5, 6, 7]).

In numerical computations, laminated glass may be modeled with layered shell elements that take into account the competing stiffness between glass and interlayer, but the libraries of most commercial numerical codes do not contain such elements; on the other hand, a full three-dimensional analysis is complicated and time consuming. This is why, in the practice and especially in the preliminary design, it is useful to rely upon simple methods. The most common approach consists in defining the so called *effective thickness*, i.e., the thickness of a glass monolith with bending properties equivalent to the laminated element. More precisely, the effective thickness of a laminated glass plate is the (constant) thickness of a monolithic plate that, under the same boundary and load conditions, presents the same maximal stress or maximal deflection. This is a very practical definition, but the literature and the technical standards record various conflicting formulas for its quantification.

The most used formulations are the one prescribed by the European Project Norm prEN-13474 [8] and that recorded in ASTM E1300 [9] following the proposal by Bennison *et al.* [10, 11] and the original work by Wölfel [12]. The first formulation condenses all the effects of the interlayer in one coefficient ω provinding a linear interpolation between the layered and the monolithic limit; however, as it will be demonstrated later on in Section 4.2, this method is quite crude and leads to contradictory results. On the other hand the method of ASTM E1300, as discussed in [13, 14], gives excellent results for laminated beams under uniformly distributed loading, but is inaccurate in other cases. Very recently, an alternative method [13], called *Enhanced Effective Thickness*, has been proposed by the authors. This is based upon a variational approach where, through minimization of the strain energy functional, the best approximation for the response of laminated glass is selected among a restricted class of shape functions for the deflection surface. The main underlying hypotheses are: *i)* the interlayer has no axial or bending stiffness, but only shear stiffness; *ii)* shear strain of glass is negligible; *iii)* both glass and polymer are linear elastic materials; *iv)* geometric

non-linearities are ineffective. Remarkably the method, originally conceived of for beams under bending [13], can be naturally extended to the two-dimensional case of plates [14] under the most various load and boundary conditions.

In general, all the aforementioned methods have been formulated for laminates with two glass layers and one interlayer. Attempts have been made to extend the prEN-13474 and the ASTM E1300 methods to the case of three of more glass plies, but the accuracy is in general not satisfactory, as it will be shown later on in Section 4. The purpose of this Article is to show that the Enhanced Effective Thickness method can be naturally extended to the case of multilayered laminated glass beams. More specifically, the cases that will be treated here are laminated beams composed either by three glass layers of arbitrary thickness, or by an arbitrary number of equally-thick glass layers. By defining an effective moment of inertia of the composed beam as the weighted *harmonic* mean of the moments of inertia corresponding to the layered and monolithic limit, practical formulas for the *stress-* and *deflection-effective thickness* are proposed. The method covers various boundary and loading conditions. Comparisons with numerical experiments highlight the much higher accuracy of the proposed approach with respect to the other formulations.

2   Five-layered beams with glass plies of arbitrary thickness.

With respect to a reference system (x, y) as in Fig. 1, consider the laminated beam of length l and width b, composed of three glass plies, of thickness h_1, h_2 and h_3 and Young's mudulus E, bonded by thin polymeric interlayers (of thickness t_1 and t_2 respectively), whose shear modulus is denoted by G. The beam is loaded by an arbitrary load per unit length $p(x)$,

not necessarily uniformly distributed. Let

$$A_i = h_i b, \ I_i = \frac{b h_i^3}{12} \ (i = 1, .., 3), \ H_1 = t_1 + \frac{h_1 + h_2}{2}, \ H_2 = t_2 + \frac{h_2 + h_3}{2}. \qquad (2.1)$$
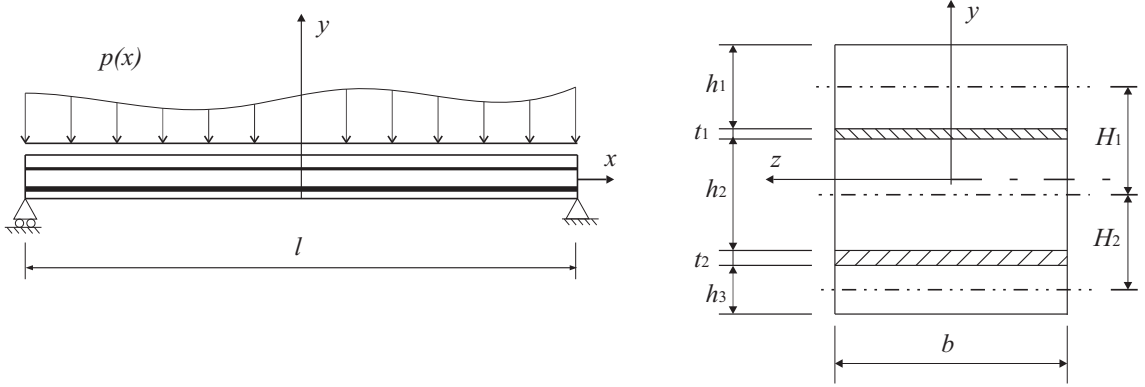

Figure 1: Five-layered simply-supported laminated glass beam composed of three glass plies bonded by polymeric interlayers. Longitudinal view and cross section (not in the same scale).

2.1   The model

Under the hypotheses that glass-polymer bond is perfect and the interlayer strain in direction y is negligible, provided that strains are small and rotations moderate, the kinematics is completely described by the vertical displacement $v(x)$, the same for the three glass components, and the horizontal displacements $u_1(x)$, $u_2(x)$ and $u_3(x)$ of the centroid of the cross-sectional areas of glass plies. Following the same procedure of [13], the shear strain in the two interlayers is given by

$$\gamma_1(x) = \frac{1}{t_1}[u_1(x) - u_2(x) + v'(x)H_1], \quad \gamma_2(x) = \frac{1}{t_2}[u_2(x) - u_3(x) + v'(x)H_2]. \qquad (2.2)$$

The total strain energy of the laminated glass beam [15] is provided by the flexural and extensional contributions of the three glass layers, by the part corresponding to the shear deformation of the interlayers, by the work of the external loads $p(x)$, and reads

$$\mathfrak{E}[u_1(x), u_2(x), u_3(x), v(x)] =$$

$$\int_{-l/2}^{l/2} \left\{ \frac{1}{2} \Big[E(I_1 + I_2 + I_3)v''(x)^2 + E[A_1 u_1'(x)^2 + A_2 u_2'(x)^2 + A_3 u_3'(x)^2] + \right.$$

$$\left. + Gb\Big[\frac{1}{t_1}(u_1(x) - u_2(x) + v'(x)H_1)^2 + \frac{1}{t_1}(u_2(x) - u_3(x) + v'(x)H_2)^2\Big]\Big] + p(x)v(x) \right\} dx \,.$$

$$(2.3)$$

The first variation of this functional with respect to the variables $v(x)$ and $u_i(x)$, $i = 1, .., 3$, gives respectively the Euler-Lagrange equilibrium equations

$$E(I_1 + I_2 + I_3)v''''(x) - Gb\Big[\frac{H_1}{t_1}(u_1(x) - u_2(x) + v'(x)H_1)' + \frac{H_2}{t_2}(u_2(x) - u_3(x) + v'(x)H_2)'\Big] + p(x) = 0 \,,$$

$$(2.4)$$

$$EA_1 u_1''(x) - \frac{Gb}{t_1}(u_1(x) - u_2(x) + v'(x)H_1) = 0 \,, \qquad (2.5)$$

$$EA_2 u_2''(x) - Gb\Big[-\frac{1}{t_1}(u_1(x) - u_2(x) + v'(x)H_1) + \frac{1}{t_2}(u_2(x) - u_3(x) + v'(x)H_2)\Big] = 0 \,, \quad (2.6)$$

$$EA_3 u_3''(x) + \frac{Gb}{t_2}(u_2(x) - u_3(x) + v'(x)H_2) = 0 \,. \qquad (2.7)$$

Recalling that $EA_i u_i''(x)$ is the derivative of the axial force $N_i(x)$ in the $i-th$ glass layer, conditions (2.5), (2.6) and (2.7) represent the axial equilibrium of the glass plies under the mutual shear force per unit length $\tau_i(x) = Gb\gamma_i(x)$, transmitted by the polymeric interlayer, as schematically represented in Fig. 2.a. Moreover, from fig. 2.b, it can be observed that the shear coupling offered by the upper interlayer is statically equipollent to a distributed torque per unit length in the upper glass ply (layer 1) equal to $-(h_1/2 + t_1^*)\tau_1(x)$, where t_1^* is arbitrary, and equal to $-(h_2/2 + t_1 - t_1^*)\tau_1(x)$ in the middle ply (layer 2). Similarly, the shear stress in the lower interlayer gives in the middle glass ply (layer 2) the distributed torque per unit length $-(h_2/2 + t_2^*)\tau_2(x)$, with t_2^* arbitrary, and $-(h_3/2 + t_2 - t_2^*)\tau_2(x)$ in the bottom ply (layer 3). Summing up the contributions, the total torque per unit length acting on the whole layered beam is equal to

$$m(x) = -\left(\frac{h_1 + h_2}{2} + t_1\right)\tau_1(x) - \left(\frac{h_2 + h_3}{2} + t_2\right)\tau_2(x) = -H_1\tau_1(x) - H_2\tau_2(x)\,. \quad (2.8)$$

Hence, (2.4) represents the equilibrium equation of the whole laminated package under bending, due to the external load $p(x)$ and a distributed torque per unit length $m(x)$.

In conclusion, equations (2.4), (2.5), (2.6) and (2.7) may be conveniently rewritten as

$$
\begin{aligned}
&EIv''''(x) + p(x) + m'(x) = 0\,, \\
&N_1'(x) = \tau_1(x)\,, \\
&N_2'(x) = -\tau_1(x) + \tau_2(x)\,, \\
&N_3'(x) = -\tau_2(x)\,.
\end{aligned}
\qquad (2.9)
$$

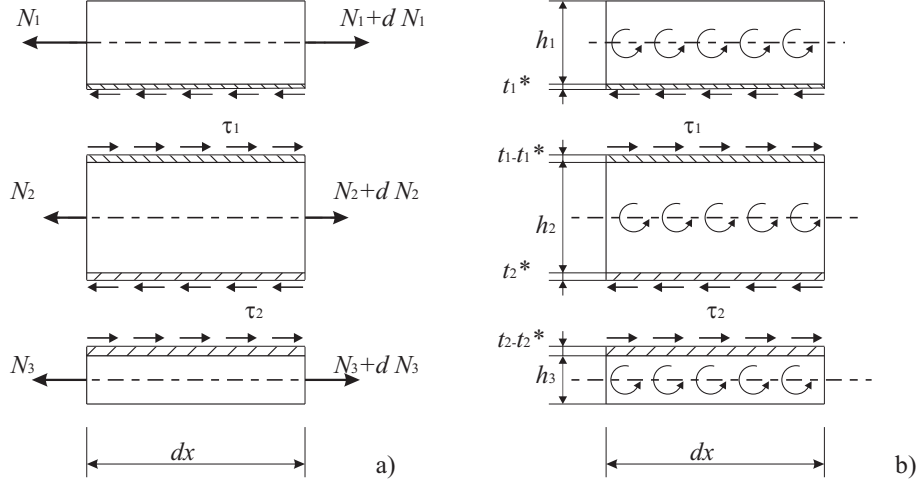Standard arguments in the calculus of variation [16] furnish the boundary conditions

Figure 2: Physical interpretation of Euler-Lagrange equations.

$$\left[-\Big(E(I_1 + I_2 + I_3)v'''(x) + Gb(\gamma_1 H_1 + \gamma_2 H_2)\Big)\delta v(x)\right]_{-l/2}^{l/2} = 0\,,$$

$$\left[E(I_1 + I_2 + I_3)v''(x)\delta v'(x)\right]_{-l/2}^{l/2} = 0\,,$$

$$\left[EA_1 u_1'(x)\delta u_1(x)\right]_{-l/2}^{l/2} = 0\,, \tag{2.10}$$

$$\left[EA_2 u_2'(x)\delta u_2(x)\right]_{-l/2}^{l/2} = 0\,,$$

$$\left[EA_3 u_3'(x)\delta u_2(x)\right]_{-l/2}^{l/2} = 0\,,$$

where $\delta v(x)$, $\delta u_i(x)$ $(i = 1, .., 3)$ denote the variations of $v(x)$ and $u_i(x)$, respectively. The meaning of (2.10) is that, in order to rended each term equal to zero, at the boundary points $x = \pm l/2$ one can either prescribe the value of the unknown field, so that the corresponding variation is null (geometric boundary condition); or when the variation is arbitrary because the value at the boundary is not prescribed, the corresponding coefficient must be zero (natural boundary condition). For example, if the $i - th$ glass layer the beam is not constrained at, say, $x = l/2$, so that $\delta u_i(l/2) \neq 0$, then the correspondent axial force $EA_i u_i'(l/2)$ must be null.

A relationship between the horizontal displacement of the glass plies can be found by

rearranging equations (2.5), (2.6) and (2.7) in the form

$$(EA_1 u_1'(x) + EA_2 u_2'(x) + EA_3 u_3'(x))' = 0$$

$$\Rightarrow \quad EA_1 u_1'(x) + EA_2 u_2'(x) + EA_3 u_3'(x) = N_1(x) + N_2(x) + N_3(x) = const =: N, \quad (2.11)$$

where N represents the constant resultant axial force. This finding is not surprising because no axially oriented force is supposed to act on the beam.

If the axial displacement of the glass plies is not constrained at one of extremities, say $x = -l/2$, so that $\delta u_i(-l/2) \neq 0$ and from $(2.10)_{3-4-5}$ $EA_i u_i'(-l/2) = 0$ $(i = 1, .., 3)$, equation (2.11) leads to $N_1(x) + N_2(x) + N_3(x) = 0$, $\forall x \in (-l/2, l/2)$. In the most general case in which the beam is constrained at both its ends so that $\delta u_i(\pm l/2) = 0$, we suppose that no overall axial elongation is given to each glass ply, i.e., $\int_{-l/2}^{l/2} u_i'(x) dx = 0$ $(i = 1, .., 3)$. This allows to evaluate from (2.11)

$$0 = \int_{-l/2}^{l/2} (EA_1 u_1'(x) + EA_2 u_2'(x) + EA_3 u_3'(x)) dx = \int_{-l/2}^{l/2} N \, dx = Nl \quad \Rightarrow \quad N = 0. \quad (2.12)$$

In conclusion, in both cases the following relation holds:

$$N_1(x) + N_2(x) + N_3(x) = (EA_1 u_1(x) + EA_2 u_2(x) + EA_3 u_3(x))' = 0$$

$$\Rightarrow \quad A_1 u_1(x) + A_2 u_2(x) + A_3 u_3(x) = const. \quad (2.13)$$

In order to prevent the rigid body motion in x direction, the horizontal displacement has to be constrained. Since this constrain is quite arbitrary, we suppose that it is such that

$$A_1 u_1(x) + A_2 u_2(x) + A_3 u_3(x) = 0\,. \tag{2.14}$$

In other words, there is always a horizontal rigid translation for which the constant in (2.13) is null. Condition (2.14) allows noteworthy simplifications in the following analysis.

2.2   Layered and monolithic limits

It should be observed that whenever $G \to 0$, the Euler's equations (2.4), (2.5), (2.6) and (2.7) become

$$\begin{cases} E(I_1 + I_2 + I_3)v''''(x) + p(x) = 0\,, \\ EA_1 u_1''(x) = 0\,, \\ EA_2 u_2''(x) = 0\,, \\ EA_3 u_3''(x) = 0\,. \end{cases} \tag{2.15}$$

It can be recognized that these correspond to the the equilibrium of three frictionless sliding glass beams. This case is the *layered limit*.

In the case that $G \to +\infty$, a relationship between horizontal and vertical displacements can be obtained by imposing that $\gamma_1(x) = 0$ and $\gamma_2(x) = 0$. Using (2.2) and adding condition (2.14), a system of equations is obtained in the form

$$\begin{cases} h_1 u_1(x) + h_2 u_2(x) + h_3 u_3(x) = 0, \\ u_1(x) - u_2(x) + v'(x)H_1 = 0, \\ u_2(x) - u_3(x) + v'(x)H_2 = 0, \end{cases} \Rightarrow \begin{cases} u_1(x) = -\frac{h_2 H_1 + h_3(H_1 + H_2)}{h_1 + h_2 + h_3}v'(x)\,, \\ u_2(x) = \frac{h_1 H_1 - h_3 H_2}{h_1 + h_2 + h_3}v'(x)\,, \\ u_3(x) = \frac{h_2 H_2 + h_1(H_1 + H_2)}{h_1 + h_2 + h_3}v'(x)\,. \end{cases} \tag{2.16}$$

Denoting with d_i $(i = 1, .., 3)$ the distance with sign (positive in the direction of increasing y) of the mid-plane of the i-th glass layer from the mid-plane of the whole laminated glass beam (shown in figure 1), it can be easily found that

$$d_1 = \frac{h_2 H_1 + h_3(H_1 + H_2)}{h_1 + h_2 + h_3} \,,$$

$$d_2 = -\frac{h_1 H_1 - h_3 H_2}{h_1 + h_2 + h_3} \,,$$

(2.17)

$$d_3 = -\frac{h_2 H_2 + h_1(H_1 + H_2)}{h_1 + h_2 + h_3} \,.$$

Hence, relations (2.16) may be rearranged to give

$$u_i(x) = -d_i v'(x) \,, \quad i = 1, ..3.$$

(2.18)

Then, using the first and fourth equation of (2.9), the shear force per unit length transmitted by the interlayer can be simply written as

$$\begin{cases} b\,\tau_1(x) = E A_1 u_1''(x) = -E A_1 d_1 v'''(x) \,, \\ b\,\tau_2(x) = -E A_3 u_3''(x) = E A_3 d_3 v'''(x) \,, \end{cases}$$

(2.19)

while in the first relationship of (2.9) the term $m'(x)$ can be rewritten as

$$m'(x) = -b\left[\tau_1(x)H_1 + \tau_2(x)H_2\right]' = E(A_1 d_1 H_1 - A_3 d_3 H_3)v''''(x) = E(A_1 d_1^2 + A_2 d_2^2 + A_3 d_3^2)v''''(x) \,.$$

(2.20)

In conclusion, the relevant equations of the system (2.9) reduce to

$$\begin{cases} EI_{tot}v''''(x) + p(x) = 0\,, \\[2mm] b\,\tau_1(x) = -EA_1 d_1 v'''(x)\,, \\[2mm] b\,\tau_2(x) = EA_3 d_3 v'''(x)\,, \end{cases} \qquad (2.21)$$

where I_{tot} is the moment of inertia associated with the monolithic limit, given by

$$I_{tot} = I_1 + I_2 + I_3 + A_1 d_1^2 + A_2 d_2^2 + A_3 d_3^2. \qquad (2.22)$$

This expression is quite important because, in general, the monolithic limit is erroneously associated with a ply whose thickness is the sum of the thicknesses of the glass plies; on the other hand, (2.22) gives the inertia of the glass layers properly spaced of the interlayer gaps t_1 and t_2 (Figure 1). The last two equations in (2.21) are the shear stress in the interlayers, calculated according with Jourawski's formulation.

2.3   The Enhanced Effective Thickness (EET) approach

Once external load $p(x)$ and boundary conditions are given, the fields $v(x)$ and $u_i(x)$ can be determined by integrating the differential system (2.4), (2.5), (2.6) and (2.7), with boundary conditions derived from (2.10). An approximation can be found by choosing an appropriate class of shape functions for the unknown fields $v(x)$ and $u_i(x)$, defined up to a few parameters that will be determined from energy minimization. This procedure has already been used in [13] for the case of three-layered beams.

The shape functions for the vertical displacement $v(x)$ and for the horizontal displacements $u_i(x)$ ($i = 1, ..., 3$), must be compatible with the qualitative properties of the exact solution and, in particular, must comprehend the monolithic-limit solution, when $G \to \infty$,

and the layered-limit solution, when $G \to 0$. In terms of the field $v(x)$, such borderline cases correspond, respectively, to the fields $v_M(x)$ and $v_L(x)$ that, being the solutions of the differential equations

$$EI_{tot}v_M''''(x) + p(x) = 0 \,, \quad E(I_1 + I_2 + I_3)v_L''''(x) + p(x) = 0 \,, \qquad (2.23)$$

have to be of the form

$$v_M(x) = -\frac{g(x)}{EI_{tot}} \,, \quad v_L(x) = -\frac{g(x)}{E(I_1 + I_2 + I_3)} \,, \qquad (2.24)$$

where $g(x)$ is a function that is uniquely determined from the form of the external load and the geometric boundary conditions of the beam. It is thus natural to think of an approximating class of solutions in the form

$$v(x) = -\frac{g(x)}{EI_R} \,, \qquad (2.25)$$

where $g(x)$ plays the role of a shape function for the vertical displacement and I_R is the effective moment of inertia of the laminate glass beam, comprised between the values $(I_1 + I_2 + I_3)$, corresponding to the layered limit, and I_{tot}, associated with the monolithic limit. It is convenient to define I_R as the weighted harmonic mean of this two values through the parameter η, a non-dimensional quantity tuning the plate response from the layered limit $(\eta = 0)$ to the monolithic limit $(\eta = 1)$. In conclusion, we can write

$$\frac{1}{I_R} = \frac{\eta}{I_{tot}} + \frac{1 - \eta}{I_1 + I_2 + I_3} \,. \qquad (2.26)$$

For what the horizontal displacement is concerned, consider first the layered and mono-lithic limit. In the *layered limit*, one finds from the last three equations of (2.15) that the horizontal displacement, say $u_{iL}(x)$ $(i = 1, .., 3)$ to distinguish, is a linear function. But if we suppose, as done to derive (2.12), that the overall axial elongation in each glass ply is zero, it can be easily concluded that $u_{iL} = 0$ $(i = 1, .., 3)$, which is also compatible with condition (2.14). In the *monolithic* limit, the horizontal displacements, say $u_{iM}(x)$ $(i = 1, .., 3)$, are related with the vertical component $v(x)$ through equations (2.18). In conclusion, one has

$$u_{iL} = 0, \quad u_{iM} = -d_i v'(x) = d_i \frac{g'(x)}{EI_{tot}}, \quad i = 1, .., 3, \tag{2.27}$$

so that an approximate solution for the horizontal displacement can be sought in the form

$$u_i(x) = \beta \, d_i \frac{g'(x)}{EI_{tot}}, \tag{2.28}$$

where β is another nondimensional parameter, again tuning the response from the layered limit ($\beta = 0$), implying null horizontal force in the glass layers, to the monolithic limit ($\beta = 1$), leading to $\gamma_i(x) = 0$, $i = 1, 2$.

The corresponding strain energy (2.3) can thus be re-written as a function of the shape function and the parameters η and β in the form

$$\mathfrak{E}[u_1(x), u_2(x), u_3(x), v(x)] = \widehat{\mathfrak{E}}[\eta, \beta, g(x)]$$

$$= \int_{-l/2}^{l/2} \left\{ \frac{(I_1 + I_2 + I_3)}{2E} \left[\frac{\eta}{I_{tot}} + \frac{1 - \eta}{I_1 + I_2 + I_3} \right]^2 [g''(x)]^2 + \frac{\beta^2}{2EI_{tot}^2} (A_1 d_1^2 + A_2 d_2^2 + A_3 d_3^2)[g''(x)]^2 \right.$$

$$+ \frac{Gb}{2E^2 I_{tot}} \left[\frac{\left(\frac{\beta(-d_1 + d_2)}{I_{tot}} - \left(\frac{\eta}{I_{tot}} + \frac{1-\eta}{I_1 + I_2 + I_3} \right) H_1 \right)^2}{t_1} + \frac{\left(\frac{\beta(-d_2 + d_3)}{I_{tot}} - \left(\frac{\eta}{I_{tot}} + \frac{1-\eta}{I_1 + I_2 + I_3} \right) H_2 \right)^2}{t_2} \right] [g'(x)]^2$$

$$\left. - \frac{p(x)}{E} \left[\frac{\eta}{I_{tot}} + \frac{1 - \eta}{I_1 + I_2 + I_3} \right] g(x) \right\} dx, \tag{2.29}$$

in which $g(x)$ is given because its depends on loading and boundary conditions for the beam. Moreover, noticing that $d_1 - d_2 = H_1$ and $d_2 - d_3 = H_2$ (figure 1), the strain energy may be simplified as

$$\mathfrak{E}[u_1(x), u_2(x), u_3(x), v(x)] = \widehat{\mathfrak{E}}[\eta, \beta, g(x)]$$

$$= \int_{-l/2}^{l/2} \left\{ \frac{(I_1 + I_2 + I_3)}{2E} \left[\frac{\eta}{I_{tot}} + \frac{1 - \eta}{I_1 + I_2 + I_3} \right]^2 [g''(x)]^2 + \frac{\beta^2}{2EI_{tot}^2} (A_1 d_1^2 + A_2 d_2^2 + A_3 d_3^2)[g''(x)]^2 \right.$$

$$\left. + \frac{Gb}{2E^2 I_{tot}} \left(\frac{H_1^2}{t_1} + \frac{H_2^2}{t_2} \right) \left[\frac{\beta}{I_{tot}} + \left(\frac{\eta}{I_{tot}} + \frac{1 - \eta}{I_1 + I_2 + I_3} \right) \right]^2 [g'(x)]^2 - \frac{p(x)}{E} \left[\frac{\eta}{I_{tot}} + \frac{1 - \eta}{I_1 + I_2 + I_3} \right] g(x) \right\} dx. \tag{2.30}$$

Minimization has to be performed with respect to the free parameters η and β only. Conditions $\frac{\partial \widehat{\mathfrak{E}}}{\partial \beta} = 0$ and $\frac{\partial \widehat{\mathfrak{E}}}{\partial \eta} = 0$ lead respectively to

$$(I_1 + I_2 + I_3)\left[\frac{\eta}{I_{tot}} + \frac{1-\eta}{I_1 + I_2}\right] \int_{-l/2}^{l/2} [g''(x)]^2 dx$$

$$-\frac{Gb}{E}\left(\frac{H_1^2}{t_1} + \frac{H_2^2}{t_2}\right)\left[\frac{\beta}{I_{tot}} - \frac{\eta}{I_{tot}} - \frac{1-\eta}{I_1 + I_2}\right] \int_{-l/2}^{l/2} [g'(x)]^2 dx - \int_{-l/2}^{l/2} p(x)g(x)dx = 0,$$

$$(2.31)$$

$$\frac{\beta}{I_{tot}}(A_1 d_1^2 + A_2 d_2^2 + A_3 d_3^2) \int_{-l/2}^{l/2} [g''(x)]^2 dx$$

$$+\frac{Gb}{E}\left(\frac{H_1^2}{t_1} + \frac{H_2^2}{t_2}\right)\left[\frac{\beta}{I_{tot}} - \frac{\eta}{I_{tot}} - \frac{1-\eta}{I_1 + I_2}\right] \int_{-l/2}^{l/2} [g'(x)]^2 dx = 0,$$

$$(2.32)$$

As done in [13], these expressions may be simplified by observing from (2.23) and (2.24) that $v(x)$ of (2.25) is the exact solution of the elastic bending of a beam with constant flexural inertia I_R under the same load $p(x)$ and boundary condition of the problem. Consider the virtual work equality for this system in which the aforementioned $v(x)$ is selected as the strain/displacement field, whereas the bending moment in equilibrium with $p(x)$ is given by $M(x) = v''(x)EI_R$. The external and internal work can be written as

$$L_{ve} = -\frac{1}{EI_R} \int_{-l/2}^{l/2} p(x)g(x)dx, \quad L_{vi} = -\int_{-l/2}^{l/2} M(x)v''(x)dx = \frac{1}{EI_R} \int_{-l/2}^{l/2} [g''(x)]^2 dx,$$

$$(2.33)$$

so that one has

$$\int_{-l/2}^{l/2} p(x)g(x)dx = \int_{-l/2}^{l/2} [g''(x)]^2 dx. \tag{2.34}$$

This condition can be used to simplify (2.31) and (2.32), yielding the noteworthy relationship

$$\eta = \beta = \frac{1}{1 + \frac{E}{GbI_{tot}\left(\frac{H_1^2}{t_1} + \frac{H_2^2}{t_2}\right)}(I_1 + I_2 + I_3)(A_1 d_1^2 + A_2 d_2^2 + A_3 d_3^2)\Psi}, \tag{2.35}$$

where the coefficient Ψ reads

$$\Psi = \frac{\int_{-l/2}^{l/2}[g''(x)]^2 dx}{\int_{-l/2}^{l/2}[g'(x)]^2 dx}. \tag{2.36}$$

Evidently, the value of Ψ depends upon the geometry, boundary and loading conditions of the beam, and it is reported in Table 1 for the most common cases. Remarkably, these data are identical to those recorded in [17] for a three-layered beam. Indeed, (3.17) represents the natural extension of the corresponding expression obtained in [13].
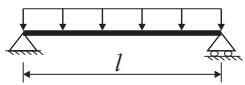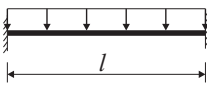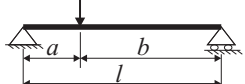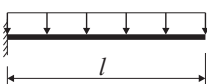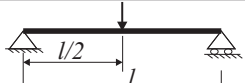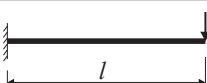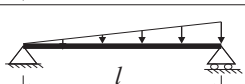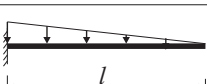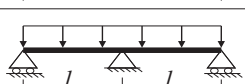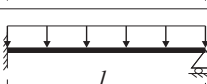
Loading and boundary conditions	Ψ	Loading and boundary conditions	Ψ
	$\frac{168}{17\,l^2}$		$\frac{42}{l^2}$
	$\frac{15}{l^2 + 2ab}$		$\frac{14}{5l^2}$
	$\frac{10}{l^2}$		$\frac{5}{2l^2}$
	$\frac{10}{l^2}$		$\frac{45}{14l^2}$
	$\frac{21}{l^2}$		$\frac{21}{l^2}$

Table 1: Values of coefficient Ψ for laminated glass beams under different boundary and load conditions.

The parameter $\eta = \beta$ can be used to define the *effective thickness* of the laminated glass beam. Recalling (2.26), the *deflection-effective thickness* $\hat{h}_w$, associated with the value η, can be written in the form

$$\frac{1}{(\hat{h}_w)^3} = \frac{\eta}{h_1^3 + h_2^3 + h_3^3 + 12(h_1 d_1^2 + h_2 d_2^2 + h_3 d_3^2)} + \frac{(1-\eta)}{(h_1^3 + h_2^3 + h_3^3)} \, . \qquad (2.37)$$

The stress-effective thickness may be defined as the (constant) thickness $\hat{h}_{i;\sigma}$ of a monolithic beam for which the maximum bending stress is equal to the maximum stresses in the $i-th$ glass layer of the laminated package. The bending stress in the $i-$th glass ply is given by

$$|\sigma_{(i)}|_{\max} = \frac{\max\limits_{x} |M(x)|}{\frac{1}{6} b \, \hat{h}_{i;\sigma}^2} = \max\limits_{x} \left| \frac{N_i(x)}{A_i} \pm \frac{M_i(x)}{I_i} \frac{h_i}{2} \right| , \qquad (2.38)$$

where $N_i(x) = EA_i u_i'(x) = \frac{A_i \beta d_1}{I_{tot}} g''(x)$ and $M_i = EI_i v''(x) = -\frac{I_i}{I_R} g''(x)$. By substituting such values into equation (2.38), one obtains

$$\frac{1}{(\hat{h}_{i;\sigma})^2} = \frac{2\eta \, |d_i|}{h_1^3 + h_2^3 + h_3^3 + 12(h_1 d_1^2 + h_2 d_2^2 + h_3 d_3^2)} + \frac{h_i}{\hat{h}_w^3} \, . \qquad (2.39)$$

The maximal stress in the $i-$th glass ply is calculated by considering a monolithic beam of thickness $\hat{h}_{i;\sigma}$ under the same loading and boundary condition of the layered beam.

3   Multi-layered beams with glass plies of equal thickness

Consider now a package composed by N glass plies of equal thickness h, bonded by $N-1$ polymeric interlayers of thickness t (Figure 3). Again, the beam length is l, its width is b, while E denotes the Young's modulus of glass and G is the shear modulus of the polymer.
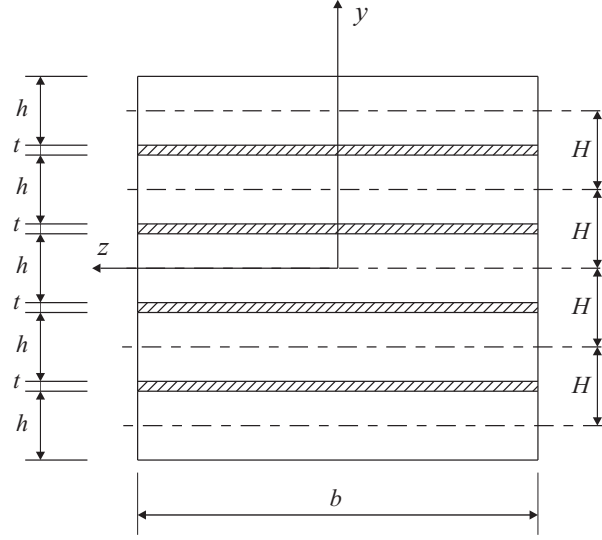
3.1   The strain energy functional

Let

Figure 3: Laminated glass package composed of N ($N = 5$ in the picture) glass plies bonded by polymeric interlayers.

$$A = hb; \ \ I = \frac{bh^3}{12}; \ \ H = t + h. \tag{3.1}$$

Under the same hypotheses of section 2, and denoting with $u_i(x)$ the horizontal displacement of the centroid of the $i-$th glass ply, the shear strain in the $i-$th interlayer may be written as $\gamma_i(x) = \frac{1}{t}[u_i(x) - u_{i+1}(x) + v'(x)H]$ $(i = 1, .., N-1)$. Analogously with equation (2.3), the strain energy of the laminated glass beam is

$$
\begin{aligned}
\mathfrak{E}[u_i(x), v(x)] \ &= \ \int_{-l/2}^{l/2} \left\{ \frac{1}{2} \left[ENIv''(x)^2 + EA\sum_{i=1}^{N} u_i'(x)^2 \right] + \right. \\
&+ \ \left. \frac{Gb}{t} \sum_{i=1}^{N-1} [u_i(x) - u_{i+1}(x) + v'(x)H]^2 + p(x)v(x) \right\} dx \, . \tag{3.2}
\end{aligned}
$$

Minimization of this functional with respect to $v(x)$ and $u_i(x)$, $(i = 1, .., N)$ leads to the $N+1$ Euler-Lagrange equations

$$ENIv''''(x) - \frac{GbH}{t} \sum_{i=1}^{N-1} [u_i(x) - u_{i+1}(x) + v'(x)H]' + p(x) = 0 \,, \tag{3.3}$$

$$EAu_1''(x) - \frac{Gb}{t}(u_1(x) - u_2(x) + v'(x)H) = 0 \,,$$
$$EAu_i''(x) + \frac{Gb}{t}[(u_{i-1}(x) - u_i(x) + v'(x)H) - (u_i(x) - u_{i+1}(x) + v'(x)H)] = 0 \,, \ i = 2,..,N-1 \,,$$
$$EAu_N''(x) + \frac{Gb}{t}(u_{N-1}(x) - u_N(x) + v'(x)H) = 0 \,, \tag{3.4}$$

with boundary conditions

$$\left[-\left(ENIv'''(x) + \frac{GbH}{t} \sum_{i=1}^{N-1} [u_i(x) - u_{i+1}(x) + v'(x)H]\right)\delta v(x) \right]_{-l/2}^{l/2} = 0 \,,$$
$$\left[ENIv''(x)\delta v'(x) \right]_{-l/2}^{l/2} = 0 \,, \tag{3.5}$$
$$\left[EAu_i'(x)\delta u_i(x) \right]_{-l/2}^{l/2} = 0 \quad \forall \ i = 1,..,N \,,$$

where $\delta v(x)$, $\delta u_i(x)$ $(i = 1,..,N)$ denote the variations of $v(x)$ and $u_i(x)$. The interpretation of these equations is the same of Section 2.

Furthermore, with the same argument that has led to (2.13), it can be demonstrate that a relationship between horizontal displacements of the glass plies exists, of the type

$$\sum_{i=1}^{N} u_i(x) = 0 \,, \tag{3.6}$$

which is of this simple form because all the glass plies have the same cross sectional area. If the glass plies were of different thickness, in (3.6) each term $u_i(x)$ would have been multiplied by the corresponding cross sectional area A_i as in (2.14), but it would be difficult then to

find simple formulas for the most general case.

3.2   Layered and monolithic limit

Whenever the layered limit is attain, the governing equilibrium equations of the laminated glass beam can be obtained simply by imposing $G = 0$ in (3.2) so that

$$\begin{cases} ENIv''''(x) + p(x) = 0 \,, \\ EAu_i''(x) = 0 \,. \quad \forall i = 1, .., N \,. \end{cases} \tag{3.7}$$

On the other hand, whenever $G \to \infty$, the shear strain in the interlayer tends to zero, i.e, $\gamma_i(x) = 0$, $i = 1, .., N - 1$, leading to

$$u_{i+1}(x) = u_i(x) + v'(x)H \,, \quad \forall i = 1, .., N - 1. \tag{3.8}$$

Henceforth, relationships between horizontal and vertical displacements may be obtain by solving the system of equations (3.8) and (3.6). One obtains

$$u_1 = \frac{2i - (N + 1)}{2}v'(x)H = -d_iv'(x)H; \tag{3.9}$$

where $d_i = \frac{(N+1)-2i}{2}H$ is again the distance with sign of the mid-plane of the $i-$th glass ply from the mid-plane of the laminated glass package, positive with increasing y (figure 3). The shear force per unit length $\tau_i(x) = Gb\gamma_i(x)$ can be evaluated by rearranging equations (3.4), to give

$$\tau_i(x) = Gb\gamma_i(x) = EA\sum_{j=1}^{i} u_j''(x) = EAv'''(x)H\sum_{j=1}^{i} \frac{2i-(N+1)}{2} = EAv'''(x)H\frac{i(i-N)}{2}\,.$$

(3.10)

Now, the second term of (3.3) can be evaluated as

$$GbH\sum_{i=1}^{N-1} \gamma_i'(x) = EAH^2v''''(x)\sum_{i=1}^{N-1} \frac{i(i-N)}{2} = -\frac{EAH^2}{12}(N-1)N(N+1)v''''(x).$$

(3.11)

By substituting such result in (3.3), the equilibrium equations for the monolithic limit turn out to be

$$EI_{tot}v''''(x) + p(x) = 0\,,$$

(3.12)

where $I_{tot} = NI + \frac{AH^2}{12}(N-1)N(N+1)$ represents the moment of inertia of a monolithic plate, whose inertia is that of the two glass plies properly spaced of the gap given by the thickness of the interlayer. It can be shown that this quantity corresponds to

$$I_{tot} = NI + A\sum_{i=1}^{N} d_i^2 = NI + A\sum_{i=1}^{N} \left(\frac{(N+1)-2i}{2}H\right)^2 = NI + \frac{AH^2}{12}(N-1)N(N+1)\,,$$

(3.13)

which represents the analogous of (2.22).

3.3   The EET approach

Following the same procedure of Section 2.3, the solution for the vertical displacement can be found in the form (2.25), where $g(x)$ is a shape function, chosen according to the loading and boundary condition, and the *effective* moment of inertia of the beam is defined as

$$\frac{1}{I_R} = \frac{\eta}{I_{tot}} + \frac{1-\eta}{N \cdot I}.$$ (3.14)

The shape function for the horizontal displacement can be found of the same form (2.28). Under these assumption, since $d_i - d_{i+1} = H$, the shear strain in the ith interlayer is

$$\gamma_i = \frac{H}{Et}\left[\frac{\beta}{I_{tot}} - \left(\frac{\eta}{I_{tot}} + \frac{1-\eta}{NI}\right)\right]g'(x), \quad i = 1,..,N-1.$$ (3.15)

Henceforth, the strain energy (3.2) may be written as

$$
\begin{aligned}
\mathfrak{E}[u_i(x), v(x)] = \widehat{\mathfrak{E}}[\eta, \beta, g(x)] = \\
= \int_{-l/2}^{l/2} \Bigg\{ \frac{NI}{2E}\left[\frac{\eta}{I_{tot}} + \frac{1-\eta}{NI}\right]^2 [g''(x)]^2 + \frac{\beta^2}{2EI_{tot}^2}\frac{AH^2(N-1)N(N+1)}{12}[g''(x)]^2 + \\
+ \ \frac{GbH^2(N-1)}{2tE^2}\left[\frac{\beta}{I_{tot}} - \left(\frac{\eta}{I_{tot}} + \frac{1-\eta}{NI}\right)\right]^2 g[g'(x)]^2 - \frac{p(x)}{E}\left[\frac{\eta}{I_{tot}} + \frac{1-\eta}{NI}\right]g(x) \Bigg\} dx,
\end{aligned}
$$ (3.16)

where, we recall, $g(x)$ is known.

Analogously to Section 2.3, the minimization of such energy with respect to the parameter η and β leads to

$$\eta = \beta = \frac{1}{1 + \frac{Et}{Gb}\frac{N^2 IA(N+1)}{I_{tot}}\Psi}, \tag{3.17}$$

where the coefficient Ψ is again given by (2.36) and recorded in Table 1 for several cases of relevant importance. Finally, in analogy with equations (2.37) and (2.39), it can be shown that the deflection- and stress-effective thickness can be defined as

$$\frac{1}{(\hat{h}_w)^3} = \frac{\eta}{Nh^3 + hH^2(N-1)N(N+1)} + \frac{(1-\eta)}{Nh^3}, \tag{3.18}$$

$$\frac{1}{(\hat{h}_{i;\sigma})^2} = \frac{2\eta\,|d_i|}{Nh^3 + hH^2(N-1)N(N+1)} + \frac{h}{\hat{h}_w^3}. \tag{3.19}$$

It is evident that the maximum stress occurs in the external glass plies, i.e. for $i=1$ and $i=N$. Since the absolute value of the distance of the mid-plane from the mid-plane of the laminated package is $\frac{N+1}{2}H$, the maximum bending stress in the laminated package may be calculated by using the stress-effective thickness of the external plies

$$\frac{1}{(\hat{h}_{\sigma EXT})^2} = \frac{\eta\,(N+1)H}{Nh^3 + hH^2(N-1)N(N+1)} + \frac{h}{\hat{h}_w^3}. \tag{3.20}$$

The conclusion is that the EET approach can be easily extended to the case of multi-layered laminated beams, with no particular difficulties.

4 Comparison with other methods

Diverse alternative formulations have been proposed to calculate the effective thickness of laminated glass. Although having been conceived for a composite package of two glass

layers and one interlayer, their extrapolation to the case of multi-layered laminated glass has been attempted. The aim of this Section is to compare results from these formulations with those obtainable with the EET method.

4.1   Iterative Wölfel- Bennison method according to ASTM-E1300

The formulation based upon the original approach by Wölfel [12], later developed by Bennison *et al.* [10, 18] and mentioned in the American standard ASTM-E1300 [9], relies upon several simplifying assumptions that, as discussed at length in [13], render it accurate for the case of simply supported beams under uniformly distributed loading.

With the same notation of (2.1), the method defines an *equivalent moment of inertia I_{eq}* of a laminated glass beam, of length l and width b, composed by *two* glass plies of thickness h_1 and h_2 bonded by *one* polymeric interlayer of thickness t, as the arithmetic mean of the inertiae of monolithic and layered limit. If $A_i = bh_i$, $I_i = bh_i/12$, $H = (h_1 + h_2)/2 + t$, then I_{eq} reads

$$I_{eq} = I_1 + I_2 + \Gamma \frac{A_1 A_2}{A_1 + A_2} H^2 \,, \tag{4.1}$$

where the shear transfer coefficient Γ is given by

$$\Gamma = \frac{1}{1 + \beta \frac{tE}{Gbl^2} \frac{A_1 A_2}{A_1 + A_2}} \,. \tag{4.2}$$

In particular, $\Gamma = 0$ corresponds to the layered limit, whereas $\Gamma = 1$ to perfect bonding. The coefficient β depends upon load and boundary conditions and, for the most common cases, the corresponding values are recorded in [12] and [13]. Bennison *et al.* [10, 18] have proposed to use $\beta = 9.6$ as in the case of uniformly distributed loading.

From (4.1), it is immediate to evaluate the deflection-effective thickness

$$h_{ef;w} = \sqrt[3]{h_1^3 + h_2^3 + 12\Gamma \frac{h_1 h_2}{h_1 + h_2} H^2}. \tag{4.3}$$

The maximum stress in glass can be estimated through the stress-effective thicknesses

$$h_{1;ef;\sigma} = \sqrt{\frac{h_{ef;w}^3}{h_1 + 2\Gamma h_{s;2}}}, \; h_{2;ef;\sigma} = \sqrt{\frac{h_{ef;w}^3}{h_2 + 2\Gamma h_{s;1}}}, \tag{4.4}$$

where $h_{s;i} = \frac{h_i H}{h_1 + h_2}$.

As mentioned in [13], Wölfel-Bennison approach is accurate only for simply supported beams under uniformly distributed load and in those cases where the deflection curve is similar in type to this reference-case. Besides, it has been verified in [14] that, when applied to a two-dimensional plate, the method is reliable only when the deformed surface is cylindrical, so that the plate response is similar to that of a beam.

For the calculation of the effective thickness of multi-layered laminated glass, a rule of thumb, also suggested in an preliminary version of the project of European Standard prEN 13474 [8], indicates to iterate the procedure. In terms of deflection, one can calculate the effective thickness of the first and second glass layer ($h_{ef;w;12}$); from this, the effective thickness of the package composed by the equivalent monolith just determined and the third glass layer ($h_{ef;w;123}$), and so on.

In terms of stress, the procedure is not without ambiguity. In fact, one may calculate the stress-effective thickness of the first two glass layers ($h_{1;ef;\sigma}, h_{2;ef;\sigma}$), but for the second step it is not clear whether to use $h_{1;ef;\sigma}$ or $h_{2;ef;\sigma}$ for the combination with the successive glass ply; in general, it should be safe to use the minor between these two values, and to maintain this choice also for the the successive steps. However, only the stress effective

thickness of the last glass ply at the end of the iterations is significant, because all the values obtained at the previous steps are intermediate passage of no practical utility. In particular, the stress-effective thicknesses for the internal layers can never be calculated.

It should also be remarked that when the laminated glass package is composed by glass layers (or interlayers) of different thickness, the resulting values of effective thicknesses for what both stress and deflection is concerned, depend upon which layer is chosen as layer 1", i.e., upon the order in which the iterative combination is performed. Furthermore, the iterative procedure usually requires a considerable computation.

4.2 The approach proposed by prEN-13474 and its critical discussion

In its latest version, the project of European Standard prEN 13474 [8] suggests a very simple method, which is assumed to be valid for a package of N glass layers of arbitrary thickness. With the same notation of before, the deflection- and stress-effective thicknesses may be calculated as

$$h_{ef;w} = \sqrt[3]{\sum_{i=1}^{N} h_i^3 + 12\omega \sum_{i=1}^{N} (h_i d_i^2)} \ , \quad h_{ef;\sigma;i} = \sqrt{\frac{h_{ef;w}^3}{h_i + 2\omega|d_i|}} \, , , \qquad (4.5)$$

where ω is a coefficient representing the degree of shear transfer, varying between 0 (layered limit) and 1 (monolithic limit). The value of ω is tabulated in [8], Table 11, as a function of i) the "interlayer stiffness family" to which the polymer belongs[1]; ii) the environmental temperature and characteristic duration of the applied loads. It is important to note that the shear transfer coefficient ω is independent upon the geometry, the thickness of the interlayer, the loading and boundary conditions. Furthermore, no explicit formula is provided to

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[1]The various types of polymers used as interlayers are classified in four different families according to their stiffeness.

evaluate ω when the shear modulus of the interlayer, G, is known.

In our opinion, the formulation (4.5) should be questioned on a theoretical basis.

To illustrate, reference will be made to the case of a simply supported beam under uniformly distributed loading, composed by two glass plies only of thickness h_1 and h_2 is now considered. For this case, as proved in [13], the EET method provides a very accurate solution. The the shear transfer coefficient η is given by (3.17) for H_2, I_3, $A_3 = 0$ and reads

$$\eta_{2layers} = \cfrac{1}{1 + \frac{Gb}{tE} \frac{I_1+I_2}{I_{tot}} \frac{A_1 A_2}{A_1+A_2} \Psi}, \tag{4.6}$$

where the coefficient Ψ, depending on the loading and boundary conditions, is given by (2.36) and, for the case simply supported beam under uniform load, is $\Psi = \frac{168}{17l^2}$, (for different loading and boundary conditions, values of Ψ are tabulated in 1). The deflection- and stress-effective thickness can be evaluated by setting $h_3 = 0$ into equations (2.37) and (2.39), respectively.

Now, in the prEN13474, the coefficient ω depends only on the stiffness family of the interlayer, hence on his shear modulus only. We can verify that such an assumption leads to a contradiction. In fact, consider two simply supported beams, one of length $l = 1.5$ m and the other of length $l = 3$ m. We investigate four composite packages: i) 8.8.2 ($h_1 = h_2 = 8$ mm and $t = 2 \times 0.38 = 0.76$ mm); ii) 8.8.4 ($h_1 = h_2 = 8$ mm and $t = 4 \times 0.38 = 1.52$ mm); iii) 12.12.2 ($h_1 = h_2 = 12$ mm and $t = 2 \times 0.38 = 0.76$ mm); iv) 12.12.4 ($h_1 = h_2 = 12$ mm and $t = 4 \times 0.38 = 1.52$ mm).

In the considered examples, the shear modulus of the interlayer G is varied in the range of interest for common interlayers (from 10^{-2} MPa to 10^1 MPa); consequently, the coefficient $\eta_{2Layers}$ is calculated through (4.6),together with the deflection- and stress-effective thickness. Calculate now the value of the shear transfer coefficient ω giving the same values

of deflection- and stress-effective thickness acording to (4.5). Note that, for a given values of the shear modulus G and for a given geometry, the deflection- and stress-effective thickness of the EET approach correspond to an unique value of ω.

If ω depended upon G only, and if the formulation (4.5) was correct, one should find a unique correspondence between G and ω, independently of the problem under considerations. Figure 4.a) shows instead that, in a simply supported laminated glass beam of length $l = 1.5$, for the same value of G there are different values of ω depending upon the composition of the laminated package. Figure 4.b) shows that also the length of the beam ($l = 1.5$ m or $l = 3$ m) is important. Therefore, the shear coupling depends not only upon the stiffeness of the interlayer, but also on the size and composition of the laminate.
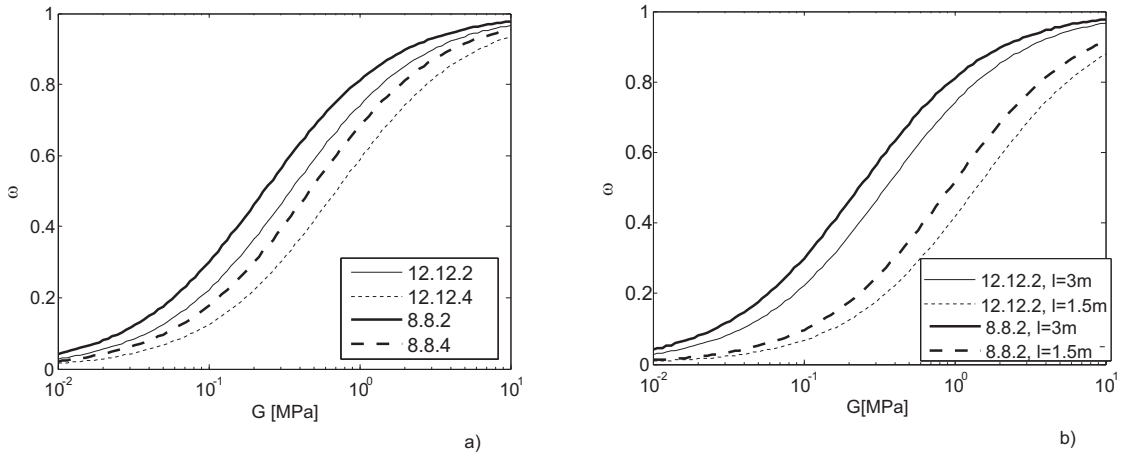

Figure 4: Relationship between the shear transfer coefficient ω and the shear modulus G of the interlayer for different values of beam length and thickness.

If the formulation proposed in the prEN13474 was correct, all the graphs would collapse in one curve. On the other hand, we notice noteworthy differences; for example, doubling of the length of the beam may increase ω by more than 100% (Figure 4.b). Indeed, we have verified in Sections 2 and 3 that the state of stress in a laminated glass beam strongly depends upon the length, composition, boundary and load conditions of the beam. This

dependence is lost in the prEn13474 approach.

4.3 Comparison with numerical experiments

The reference example will be that of a simply supported beam of length $l = 3000$ mm and width $b = 500$ mm under uniformly distributed load $p = 1$ N/mm. Other loading or boundary conditions can be analyzed with the same methods used in [13]. The Young's modulus of glass plies is $E = 70$ GPa, while the modulus G of the polymeric interlayer is varied from 10^{-2} MPa to 10^1 MPa, while its Poisson's ratio is supposed equal $\nu = 0.49$. For the sake of comparison, two different laminated glass package are here taken into account:

- three glass plies of thickness $h_1 = 5$ mm, $h_2 = 8$ mm, $h_3 = 10$ mm, bonded by two polymeric interlayer of thickness $t_1 = 0.76$ mm and $t_2 = 1.52$mm (*Package A*);

- five glass plies of the same thickness $h = 6$ mm bonded by polymeric interlayers of the same thickness $t = 0.76$ mm (*Package B*).

Numerical simulations have been made with the program Abaqus, using a 3-D mesh with solid 20-node quadratic bricks with reduced integration, available in the program library [19]. As shown in Fig. 5, the structured mesh has been crated by dividing the length of the beam in 100 elements, its width in 20 elements and the thickness of each glass layer in 3 elements.

The effective thicknesses obtained numerically are now compared with those obtainable with the EET method and with those calculated from the "Iterative" Wölfel- Bennison method and from the formulations of the the European Norm prEN-13474 (2012). For what the latter case is concerned, a correlation between G and ω is established by calculating, for each value of G, the deflection-effective thickness of a two-glass-ply laminated beam with
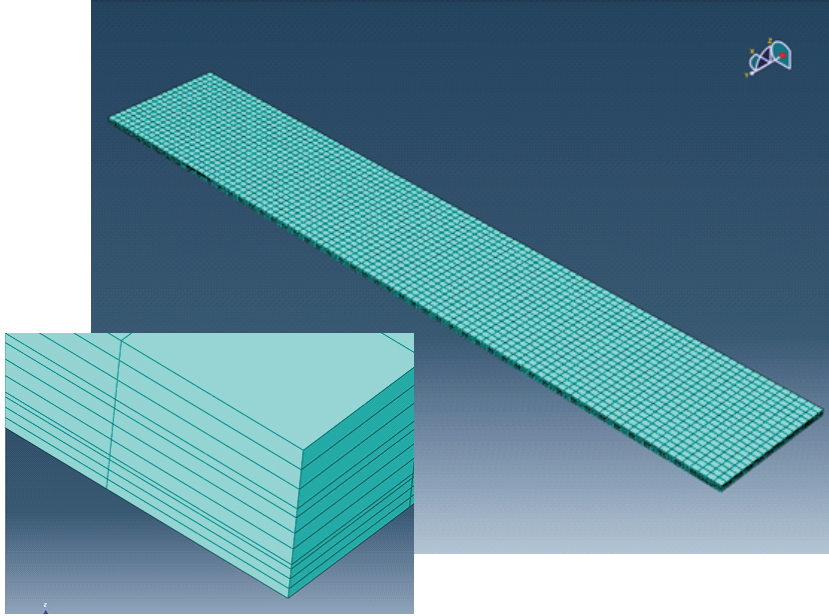
Figure 5: The mesh used in the FEM simulations for *Package A*.

the boundary and load conditions, and finding the value of ω that gives the same value through (4.5). It should be mentioned, however, that if the layers are not of the same thickness, the obtained value of ω depends upon which layers are chosen. Here, we have selected the two layers of the package that give the best approximation.

For *Package A*, Figure 6 shows the comparison of the deflection- and stress-effective thicknesses (for glass ply # 3, the one under maximum tensile stress) obtained with the Enhanced Effective Thickness approach through (2.37) and (2.39), the iterative Wölfel-Bennison approach, the method prescribed by prEN 13474 (equations 4.5), as well as the results of the numerical experiments. It is evident that the result achieved through EET and WB model are very accurate, whereas the prEN 13474 approach gives different results at the qualitative level. Most of all, the last method is not on the side of safeness, because it predicts deflection and stress values much lower than those predicted EET nd WB (higher effetive thicknesses).
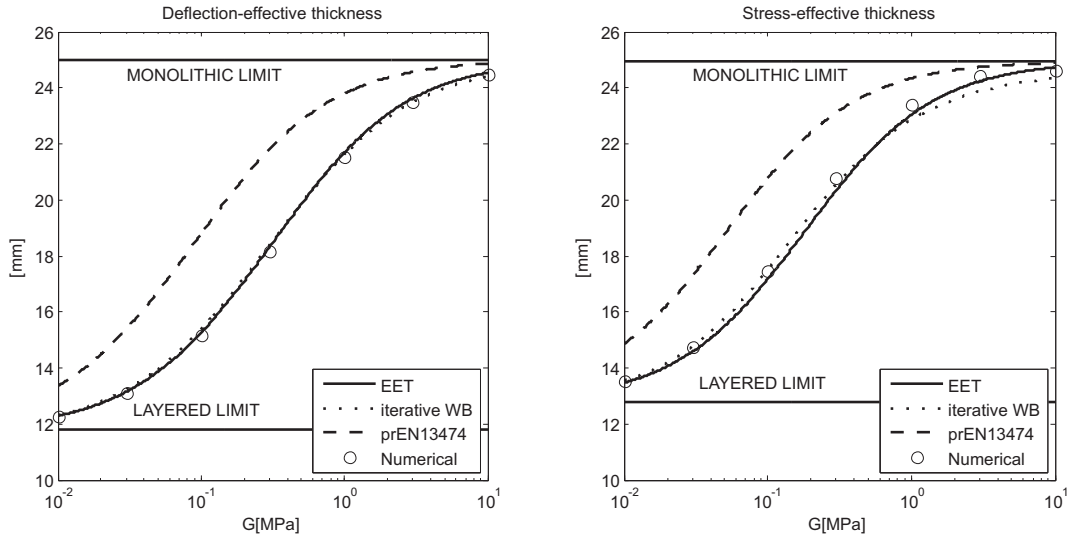
Figure 6: Simply supported beam under uniformly distributed load, *Package A*. Comparison of deflection- and stress-effective thicknesses obtained with: enhanced effective thickness approach (ETT); iterative Bennison-Wölfel approach (iterative WB); prEN 13474; numerical simulations.

For *Package B* Figure 7 compares the the deflection- and stress-effective thicknesses (of the external glass ply) obtained with the same four methods. Again, the best approximation is obtained with the EET method. The prEN13474 method is by far the less accurate.
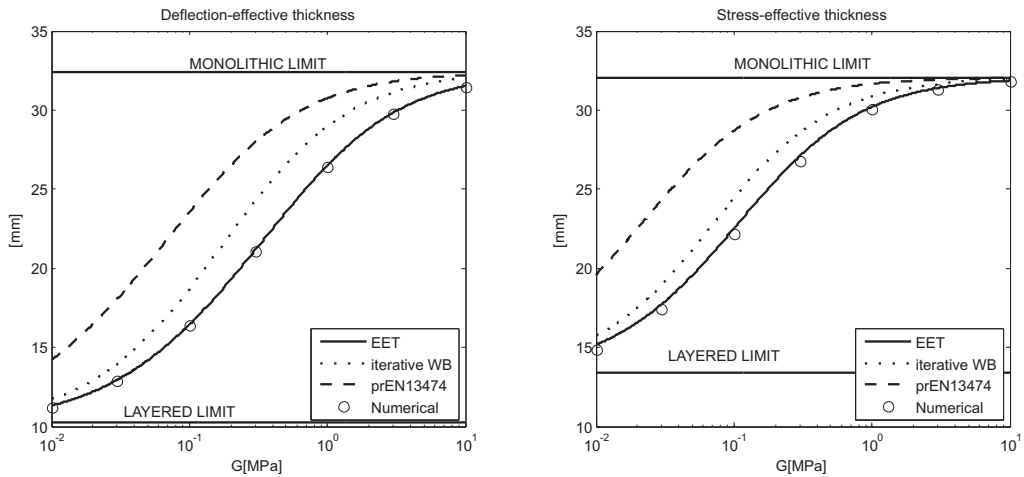

Figure 7: Simply supported beam under uniformly distributed load, *Package B*. Comparison of the effective thicknesses obtained with: the enhanced effective thickness approach (ETT); iterative Bennison- Wölfel approach (iterative WB); prEN 13474; the numerical simulations.

More examples could be made by varying the load and boundary conditions of the model problem. In general, the EET approach is the one that gives the best accuracy, especially when the boundary and load conditions are different from the case of simply supported beams under uniform loading.

5   Conclusions

The main result of this article has been to provide a natural extension of the Enhanced Effective Thickness (EET) method proposed in [13] to the case of multi-layered laminated glass beams. The formulas (2.37) and (2.39) allow to calculate the deflection- and stress-effective stiffness for a composite package of three glass layers of any thickness with no particular difficulty. If the thickness of the glass layers is constant, the expressions (3.18) and (3.19) provide a ready-to-use practical tool for the design of laminates formed by an arbitrary number of layers. The coefficients needed for these calculations are similar to those obtained for the case of a three-layered beam (two glass layers and one interlayer) and recorded in [17]. The results that can be obtained are in excellent agreement with numerical experiments, performed with accurate three-dimensional models.

Other methods that have been proposed in technical standards give in general inaccurate results when extended to the multi-layered case. For example, the iteration of the Bennison-Wölfel approach [12] [18], mentioned in the ASTM standards [9], is rather complicated, provides formulas for the calculation of the state of stress in the external glass plies only (not necessarily the most stressed) and turns out to be sufficiently accurate just for simply supported beams under uniformly distributed loads. Moreover, the result depends in general upon the order in which the iteration is performed.

The formulation adopted by the project norm prEN13474 [8] is apparently very simple

because it considers only the shear stiffness of the interlayer, but it is conceptually not justifiable. In fact, it does not take into account the important role played by load-type and size-effect (length, width and package composition) on the shear coupling that the interlayer can offer. In general, this formulation cannot be recommended in the case of three-layered beams, worst of all for multi-layered laminates.

On the contrary, the enhanced effective-thickness approach seems to represent an accurate and powerful tool for the practical calculation of laminated glass. The theory proposed here could also be applied to a package composed by an arbitrary number of layers of any thickness, even if the resulting design formulas would be slightly more complicated than those presented here. Moreover, following the same rationale, the method could be applied to the case of multilayered laminated plates, providing an extension of the theory already presented in [14].

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